

# On the power graph of a finite group

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## Abstract

The power graph  $\mathcal{P}_G$  of a finite group  $G$  is the graph with the vertex set  $G$ , where two elements are adjacent if one is a power of the other. We first show that  $\mathcal{P}_G$  has a transitive orientation, so it is a perfect graph and its core is a complete graph. Then we use the poset on all cyclic subgroups (under usual inclusion) to characterise the structure of  $\mathcal{P}_G$ . Finally, the closed formula for the metric dimension of  $\mathcal{P}_G$  is established. As an application, we compute the metric dimension of the power graph of a cyclic group.

*Key words:* group; poset; power graph; transitive orientation; comparable graph; core; resolving set; metric dimension.

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## 1 Introduction

In this paper, a graph means an undirected simple graph and a digraph means a directed graph without loops. We always use  $V(\Gamma)$  and  $E(\Gamma)$  to denote the vertex set and the edge set (resp. the arc set) of a graph (resp. digraph)  $\Gamma$ , respectively. All groups, graphs and digraphs considered are finite.

Given a group, there are different ways to associate a directed or undirected graph to the group: intersection graphs [5, 31], commuting graphs [3], prime graphs [21] and of course Cayley (di)graphs, which have a long history.

Let  $G$  be a group. The *power digraph* of  $G$  is the digraph  $\vec{\mathcal{P}}_G$  with the vertex set  $G$ , where there is an arc from  $x$  to  $y$  if  $x \neq y$  and  $y = x^m$  for some positive integer  $m$ . The *power graph*  $\mathcal{P}_G$  has the vertex set  $G$  and two distinct elements  $x$  and  $y$  are adjacent if one is a power of the other. The power digraph was introduced by Kelarev and Quinn [23, 24] and they called it directed power graph and defined it on semigroups. Motivated by this, Chakrabarty, Ghosh and Sen [9] defined power graphs of semigroups. Recently, Many interesting results on the power graphs have been obtained, see [7, 8, 11, 26, 27, 30]. In [1], Abawajy, Kelarev and Chowdhury give a survey of the current state of knowledge on this research direction by presenting all results and open questions recorded in the literature dealing with power graphs.

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Given a graph  $\Gamma$ , the digraph  $\mathcal{O}$  is an *orientation* for  $\Gamma$  if  $V(\mathcal{O}) = V(\Gamma)$  and  $|\{(u, v), (v, u)\} \cap E(\mathcal{O})| = 1$  for all  $\{u, v\} \in E(\Gamma)$ . A *transitive orientation* for  $\Gamma$  is an orientation  $\mathcal{O}$  such that  $\{(u, v), (v, w)\} \subseteq E(\mathcal{O})$  implies  $(u, w) \in E(\mathcal{O})$ . A *comparability graph* is a graph that admits a transitive orientation. It has been originally studied in [14] and characterized in [12, 15]. Recently, comparability graphs have been used to model optimization problems in railways: see [10] for a survey. Comparability graphs have an important role in graph theory because of their relationship with partially ordered sets: a comparability graph is a graph which has the vertex set a poset and join two distinct elements if they are comparable in the poset.

For a graph  $\Gamma$ , let  $d_\Gamma(u, v)$  denote the distance between two vertices  $u$  and  $v$ . By an ordered set of vertices, we mean a set  $W = \{w_1, \dots, w_k\}$  on which the ordering  $(w_1, \dots, w_k)$  has been imposed. For an ordered subset  $W = \{w_1, \dots, w_k\}$ , write  $\mathcal{D}_\Gamma(v|W) = (d_\Gamma(v, w_1), \dots, d_\Gamma(v, w_k))$ . A *resolving set* of  $\Gamma$  is an ordered subset of vertices  $W$  such that  $\mathcal{D}_\Gamma(u|W) = \mathcal{D}_\Gamma(v|W)$  if and only if  $u = v$ . The *metric dimension* of  $\Gamma$ , denoted by  $\dim(\Gamma)$ , is the minimum cardinality of a resolving set of  $\Gamma$ . Metric dimension was first introduced in the 1970s, independently by Harary and Melter [19] and by Slater [29]. It is a parameter that has appeared in various applications (see [2, 6] for more information). It was noted in [13, p. 204] and [25] that determining the metric dimension of a graph is an NP-complete problem.

In this paper, we study the power graph of a group  $G$ . In Section 2, we first construct a transitive orientation for  $\mathcal{P}_G$ , then get some properties of  $\mathcal{P}_G$ , and finally characterise the structure of  $\mathcal{P}_G$  by using the poset on all cyclic subgroups of  $G$  (under usual inclusion). In Section 3, we establish a closed formula for the metric dimension of  $\mathcal{P}_G$ .

## 2 Properties and characterization

In this section, we get some properties of the power graph of a group  $G$  and characterize the structure of  $\mathcal{P}_G$ . In Subsection 2.1, we construct a transitive orientation for  $\mathcal{P}_G$ , which is a subdigraph of the power digraph  $\vec{\mathcal{P}}_G$ . Therefore, we know that  $\mathcal{P}_G$  is a comparability graph. Then we show that it is a perfect graph and its core is complete. Since a transitive orientation uniquely determine a partially ordered set (or poset for simplify), Subsection 2.2 reviews some definitions or properties associated with posets. In Subsection 2.3, we characterize the structure of  $\mathcal{P}_G$  by using the poset on all cyclic subgroups of  $G$  (under usual inclusion).

### 2.1 Transitive orientations and comparability graphs

Let  $G$  be a group. For  $x \in G$ , denote by  $[x]$  the set of all generators of the cyclic subgroup  $\langle x \rangle$ . Write

$$\mathcal{C}'(G) = \{[x] \mid x \in G\} = \{[x_{1,1}], \dots, [x_{k,1}]\}, \text{ where } [x_{i,1}] = \{x_{i,1}, \dots, x_{i,s_i}\}. \quad (1)$$

We impose an ordering  $(x_{i,1}, \dots, x_{i,s_i})$  on the set  $[x_{i,1}]$  for each  $i \in \{1, \dots, k\}$ .

**Definition 2.1** For elements  $x$  and  $y$  in a group  $G$ , define  $x \prec y$  if one of the followings holds.

- (i) For some  $i$ ,  $x = x_{i,l}$ ,  $y = x_{i,t}$  and  $l < t$ .
- (ii)  $\langle x \rangle \subsetneq \langle y \rangle$ .

Define  $x \preceq y$  if  $x \prec y$  or  $x = y$ .

The proof of the following lemma is clear from the above definition.

**Lemma 2.2** Suppose  $G$  is a group. With reference to (1), if there exist two distinct indices  $i$  and  $j$  such that  $x_{i,l_0} \prec x_{j,t_0}$  for some positive integers  $l_0$  and  $t_0$ , then  $x_{i,l} \prec x_{j,t}$  for each  $x_{i,l} \in [x_{i,1}]$  and each  $x_{j,t} \in [x_{j,1}]$ .

Define  $\mathcal{O}_G$  as the digraph with the vertex set  $G$ , and there is an arc from  $x$  to  $y$  if  $y \prec x$ . Then  $\mathcal{O}_G$  is an orientation of  $\mathcal{P}(G)$ .

**Theorem 2.3** Let  $G$  be a group. Then  $\mathcal{O}_G$  is a transitive orientation of  $\mathcal{P}_G$  and a subdigraph of  $\vec{\mathcal{P}}_G$ . Moreover, if  $\mathcal{O}'$  is a transitive orientation of  $\mathcal{P}_G$  and a subdigraph of  $\vec{\mathcal{P}}_G$ , then  $\mathcal{O}'$  and  $\mathcal{O}_G$  are isomorphic.

*Proof.* Suppose  $\{(x, y), (y, z)\} \subseteq E(\mathcal{O}_G)$ . Then  $z \prec y$  and  $y \prec x$ , and so  $\langle z \rangle \subseteq \langle y \rangle \subseteq \langle x \rangle$ . If  $\langle z \rangle \neq \langle x \rangle$ , then  $z \prec x$ , which implies that  $(x, z) \in E(\mathcal{O}_G)$ . If  $\langle z \rangle = \langle x \rangle$ , then  $[z] = [y] = [x]$ . With reference to (1), there exists an index  $i$  such that  $z = x_{i,l}$ ,  $y = x_{i,r}$ ,  $x = x_{i,t}$  and  $l < r < t$ . So  $z \prec x$  and  $(x, z) \in E(\mathcal{O}_G)$ . It follows that  $\mathcal{O}_G$  is a transitive orientation of  $\mathcal{P}_G$ . It is clear that  $\mathcal{O}_G$  is a subdigraph of  $\vec{\mathcal{P}}_G$ .

Assume that a subdigraph  $\mathcal{O}'$  of  $\vec{\mathcal{P}}_G$  is another transitive orientation of  $\mathcal{P}_G$ . With reference to (1), if  $\langle x_{i,1} \rangle \subsetneq \langle x_{j,1} \rangle$ , then  $\{x_{i,1}, x_{j,1}\} \in E(\mathcal{P}_G)$ ,  $(x_{i,1}, x_{j,1}) \in E(\vec{\mathcal{P}}_G)$  and  $(x_{j,1}, x_{i,1}) \notin E(\vec{\mathcal{P}}_G)$ , which implies that  $(x_{i,1}, x_{j,1}) \in E(\mathcal{O}')$ . Therefore, for  $[x_{i,1}] \neq [x_{j,1}]$ , we have  $(x_{i,1}, x_{j,1}) \in E(\mathcal{O}_G)$  if and only if  $(x_{i,1}, x_{j,1}) \in E(\mathcal{O}')$ . Since the induced subgraph on  $[x_{i,1}]$  of  $\mathcal{P}_G$  is a complete graph, and all transitive orientations of a fixed complete graph are isomorphic, we conclude that  $\mathcal{O}'$  and  $\mathcal{O}_G$  are isomorphic.  $\square$

The following theorem is an immediate result from Theorem 2.3.

**Theorem 2.4** The power graph of a group is a comparability graph.

For two graphs  $\Gamma$  and  $\Gamma'$ , a *homomorphism* from  $\Gamma$  to  $\Gamma'$  is a map  $f : V(\Gamma) \rightarrow V(\Gamma')$  such that  $\{f(u), f(v)\} \in E(\Gamma')$  whenever  $\{u, v\} \in E(\Gamma)$ . The *chromatic number* of  $\Gamma$ , denoted by  $\chi(\Gamma)$ , is the least value of  $k$  such that there exists a homomorphism from  $\Gamma$  to the complete graph of order  $k$ . The *clique number* of  $\Gamma$ , denoted by  $\omega(\Gamma)$ , is the maximum order of a clique in  $\Gamma$ .

A graph  $\Gamma$  is *perfect* if  $\chi(\Lambda) = \omega(\Lambda)$  for each induced subgraphs  $\Lambda$  of  $\Gamma$ . It was noted in [4, Chapter V, Theorem 17] that comparability graphs are perfect. Hence, by Theorem 2.4, we have

**Corollary 2.5** [11, Theorem 1] The power graph of a group is perfect.

An *endomorphism* of a graph  $\Gamma$  is a homomorphism from  $\Gamma$  to itself. A *core* [16] of  $\Gamma$  is a subgraph  $\Lambda$  satisfies that every endomorphism of  $\Lambda$  is an automorphism and there exists a homomorphism from  $\Gamma$  to  $\Lambda$ . Every graph has a core, which is an induced subgraph and is unique up to isomorphism [16, Lemma 6.2.2]. A graph is called a *core* if its core is itself. Godsil and Royle [17] showed that the core of a graph  $\Gamma$  is complete if and only if  $\chi(\Gamma) = \omega(\Gamma)$ .

**Observation 1** The core of any induced subgraph of a perfect graph is complete. In particular, the core of any induced subgraph of a comparability graph is complete.

**Proposition 2.6** [9, Theorem 2.12] *Let  $G$  be a group. Then  $\mathcal{P}_G$  is complete if and only if  $G$  is a cyclic group of order  $p^m$  for some prime  $p$  and nonnegative integer  $m$ .*

Combining Theorem 2.4, Observation 1 and Proposition 2.6, we get the following corollary.

**Corollary 2.7** *Given a group  $G$ , the core of any induced subgraph of  $\mathcal{P}_G$  is complete. So  $\mathcal{P}_G$  is a core if and only if  $G$  is a cyclic group of order  $p^m$  for some prime  $p$  and nonnegative integer  $m$ .*

## 2.2 Posets

A *partially ordered set* or *poset*  $P$  is an ordered pair  $(V(P), \leq_P)$ , where  $V(P)$  is a finite set, called the *vertex set* of  $P$ , and  $\leq_P$  is a reflexive, antisymmetric and transitive binary relation on  $V(P)$ . As usual, write  $x <_P y$  if  $x \leq_P y$  and  $x \neq y$ . For any subset  $S \subseteq V(P)$ , the *subposet* of  $P$  induced by  $S$ , denoted by  $P(S)$ , is a poset  $(S, \leq_{P(S)})$ , where  $x \leq_{P(S)} y$  if and only if  $x \leq_P y$ . Two elements  $x$  and  $y$  of  $V(P)$  are *comparable* if  $x \leq_P y$  or  $y \leq_P x$ , otherwise  $x$  and  $y$  are *incomparable*. The *comparability graph* of  $P$ , denoted by  $\mathcal{G}_P$ , is the graph with the vertex set  $V(P)$ , where two distinct elements are adjacent if they are comparable.

From Theorem 2.3, we get the following example.

**Example 1** Let  $G$  be a group. With reference to Definition 2.1, the ordered pair  $(G, \preceq)$  is a poset. In the remaining of this paper, we use  $L_G$  to denote this poset. The comparability graph of  $L_G$  is the power graph of a group  $G$ , i.e.,  $\mathcal{G}_{L_G} = \mathcal{P}_G$ .

A *chain* (resp. An *antichain*) in a poset  $P$  is a subset of  $V(P)$  such that all elements in this subset are pairwise comparable (resp. incomparable). A subset  $S$  of  $V(P)$  is *homogeneous* if, for any  $y \in V(P) \setminus S$ , one of the following holds:

- For all  $x \in S$ ,  $x \leq_P y$ .
- For all  $x \in S$ ,  $y \leq_P x$ .
- For all  $x \in S$ ,  $x$  and  $y$  are incomparable.

A *homogeneous chain* (resp. *antichain*) in  $P$  is a chain (resp. an antichain) that is homogeneous. A partition  $\mathcal{S}$  of  $V(P)$  is a *homogeneous partition* of  $P$  if all elements of  $\mathcal{S}$  are homogeneous subsets. Let  $\mathcal{S}$  be a homogeneous partition of  $P$ . The *quotient*

$P/\mathcal{S} = (\mathcal{S}, \leq_{P/\mathcal{S}})$ , where two subsets  $S_1, S_2 \in \mathcal{S}$  satisfies  $S_1 \leq_{P/\mathcal{S}} S_2$  if  $S_1 = S_2$  or  $x <_P y$  for each  $x \in S_1$  and each  $y \in S_2$ . Then  $P/\mathcal{S}$  is a poset.

The inverse operation of the quotient is the *lexicographical sum* [22] defined as follows. Let  $P$  be a poset and let  $\mathbb{Q}$  be a family of posets indexed by  $V(P)$ , write  $\mathbb{Q} = \{Q_x \mid x \in V(P)\}$ . The lexicographical sum of  $\mathbb{Q}$  over  $P$ , denoted by  $P[\mathbb{Q}]$ , is the poset with the vertex set  $V(P[\mathbb{Q}]) = \{(x, y) \mid x \in V(P) \text{ and } y \in V(Q_x)\}$ , where  $(x_1, y_1) \leq_{P[\mathbb{Q}]} (x_2, y_2)$  provided that either  $x_1 = x_2$  and  $y_1 \leq_{Q_{x_1}} y_2$  or  $x_1 <_P x_2$ . One can prove that this definition is well-defined. The following result is clear.

**Lemma 2.8** *Suppose that  $\mathcal{S}$  is a homogeneous partition of a poset  $P$ . Write  $R = P/\mathcal{S}$  and  $\mathbb{S} = \{P(S) \mid S \in \mathcal{S}\}$ . Then  $P$  is isomorphic to  $R[\mathbb{S}]$ .*

Recall that the poset  $L_G = (G, \preceq)$ , where  $\preceq$  is defined in Definition 2.1. The following lemma is an immediate result from Lemma 2.2.

**Lemma 2.9** *Let  $G$  be a group. With references to (1), any element  $[x]$  in  $\mathcal{C}'(G)$  is a homogeneous chain in  $L_G$ . Consequently, the set  $\mathcal{C}'(G)$  is a homogeneous partition of  $L_G$ , and so  $L_G/\mathcal{C}'(G)$  is a quotient of  $L_G$ .*

The following result gives some equivalent conditions for comparing two distinct elements in the quotient  $L_G/\mathcal{C}'(G)$ .

**Lemma 2.10** *Given a group  $G$ , let  $[x]$  and  $[y]$  be two distinct elements in  $\mathcal{C}'(G)$ . Then the following conditions are equivalent.*

- (i)  $[x] <_{L_G/\mathcal{C}'(G)} [y]$ .
- (ii)  $\langle x \rangle \subsetneq \langle y \rangle$ .
- (iii)  $(y, x) \in E(\vec{\mathcal{P}}_G)$  and  $(x, y) \notin E(\vec{\mathcal{P}}_G)$

*Proof.* If (i) holds, then  $x \prec y$ , which implies that (ii) holds by  $[x] \neq [y]$ . It is clear that (ii) and (iii) are equivalent. Suppose (ii) holds. Then, for each  $x' \in [x]$  and each  $y' \in [y]$ , we have  $\langle x' \rangle \subsetneq \langle y' \rangle$ , and so  $x' \prec y'$ . It follows that (i) holds.  $\square$

**Proposition 2.11** [8, Theorem 2] *If  $G_1$  and  $G_2$  are groups whose power graphs are isomorphic, then their power digraphs are also isomorphic.*

**Theorem 2.12** *Suppose  $G_1$  and  $G_2$  are groups. Then the followings are equivalent.*

- (i) The power graphs  $\mathcal{P}_{G_1}$  and  $\mathcal{P}_{G_2}$  are isomorphic.
- (ii) The power digraphs  $\vec{\mathcal{P}}_{G_1}$  and  $\vec{\mathcal{P}}_{G_2}$  are isomorphic.
- (iii) The transitive orientations  $\mathcal{O}_{G_1}$  and  $\mathcal{O}_{G_2}$  are isomorphic.
- (iv) The posets  $L_{G_1}$  and  $L_{G_2}$  are isomorphic.
- (v) There is an isomorphism  $\tau$  from the quotient  $L_{G_1}/\mathcal{C}'(G_1)$  to the quotient  $L_{G_2}/\mathcal{C}'(G_2)$  such that  $|\tau(S)| = |S|$  for each  $S \in \mathcal{C}'(G_1)$ .

*Proof.* Proposition 2.11 says that (i) implies (ii). Lemma 2.3 concludes that (ii) implies (iii). From the definitions, we can see that (iii) and (iv) are equivalent. By Example 1, it is clear that (iv) implies (i). It follows from Lemmas 2.8 and 2.9 that (v) implies (iv).

Suppose (ii) holds. Let  $\sigma$  be an isomorphism from  $\vec{\mathcal{P}}_{G_1}$  to  $\vec{\mathcal{P}}_{G_2}$ . For  $S \in \mathcal{C}'(G_1)$ , define  $\tau(S) = \{\sigma(x) \mid x \in S\}$ . Pick  $x \in S$ . Then

$$\begin{aligned} S = [x] &= \{x\} \cup \{y \mid \{(x, y), (y, x)\} \subseteq E(\vec{\mathcal{P}}_{G_1})\} \\ &= \{x\} \cup \{y \mid \{(\sigma(x), \sigma(y)), (\sigma(y), \sigma(x))\} \subseteq E(\vec{\mathcal{P}}_{G_2})\}, \end{aligned}$$

which implies that

$$\tau([x]) = \{\sigma(x)\} \cup \{\sigma(y) \mid \{(\sigma(x), \sigma(y)), (\sigma(y), \sigma(x))\} \subseteq E(\vec{\mathcal{P}}_{G_2})\} = [\sigma(x)] \in \mathcal{C}'(G_2).$$

Consequently, we obtain that  $\tau$  is a bijection from  $\mathcal{C}'(G_1)$  to  $\mathcal{C}'(G_2)$  such that  $|\tau(S)| = |S|$ . By Lemma 2.10, we have  $S <_{L_{G_1}/\mathcal{C}'(G_1)} S'$  if and only if  $\tau(S) <_{L_{G_2}/\mathcal{C}'(G_2)} \tau(S')$ , and so (v) holds.  $\square$

### 2.3 Characterization

In order to give the structure of power graphs, we need the definition of the generalized lexicographic product, which was first defined by Sabidussi [28]. Given a graph  $\mathcal{H}$  and a family of graphs  $\mathbb{F} = \{\mathcal{F}_v \mid v \in V(\mathcal{H})\}$ , indexed by  $V(\mathcal{H})$ , their *generalized lexicographic product*, denoted by  $\mathcal{H}[\mathbb{F}]$ , is defined as the graph with the vertex set  $V(\mathcal{H}[\mathbb{F}]) = \{(v, w) \mid v \in V(\mathcal{H}) \text{ and } w \in V(\mathcal{F}_v)\}$  and the edge set  $E(\mathcal{H}[\mathbb{F}]) = \{ \{(v_1, w_1), (v_2, w_2)\} \mid \{v_1, v_2\} \in E(\mathcal{H}), \text{ or } v_1 = v_2 \text{ and } \{w_1, w_2\} \in E(\mathcal{F}_{v_1}) \}$ .

Recall that the comparability graph of a poset  $P$ , denoted by  $\mathcal{G}_P$ , is the graph with the vertex set  $V(P)$ , where two distinct elements are adjacent if they are comparable.

**Lemma 2.13** *Given a poset  $P$ , let  $\mathbb{Q}$  be a family of posets indexed by  $V(P)$ . Suppose  $\mathbb{G}_{\mathbb{Q}}$  consists of all comparability graphs of posets in  $\mathbb{Q}$ . Then  $\mathcal{G}_{P[\mathbb{Q}]} = \mathcal{G}_P[\mathbb{G}_{\mathbb{Q}}]$ .*

*Proof.* Write  $\mathbb{Q} = \{Q_x \mid x \in V(P)\}$ . It is clear that

$$V(\mathcal{G}_{P[\mathbb{Q}]}) = \{(x, y) \mid x \in V(P), y \in V(Q_x)\} = V(\mathcal{G}_P[\mathbb{G}_{\mathbb{Q}}]).$$

Hence, it suffices to prove  $E(\mathcal{G}_{P[\mathbb{Q}]}) = E(\mathcal{G}_P[\mathbb{G}_{\mathbb{Q}}])$ .

Suppose  $\{(x_1, y_1), (x_2, y_2)\} \in E(\mathcal{G}_{P[\mathbb{Q}]})$ . Then  $(x_1, y_1)$  and  $(x_2, y_2)$  are comparable in  $P[\mathbb{Q}]$  and  $(x_1, y_1) \neq (x_2, y_2)$ . Without loss of generality, assume that  $(x_1, y_1) <_{P[\mathbb{Q}]} (x_2, y_2)$ . Hence, either  $x_1 = x_2$  and  $y_1 <_{Q_{x_1}} y_2$  or  $x_1 <_P x_2$ . If  $x_1 = x_2$  and  $y_1 <_{Q_{x_1}} y_2$ , then  $\{y_1, y_2\} \in E(Q_{x_1})$ , which implies that  $\{(x_1, y_1), (x_2, y_2)\} \in E(\mathcal{G}_P[\mathbb{G}_{\mathbb{Q}}])$ . If  $x_1 <_P x_2$ , then  $\{x_1, x_2\} \in E(\mathcal{G}_P)$ . It follows that  $\{(x_1, y_1), (x_2, y_2)\} \in E(\mathcal{G}_P[\mathbb{G}_{\mathbb{Q}}])$ . Therefore, we have  $E(\mathcal{G}_{P[\mathbb{Q}]}) \subseteq E(\mathcal{G}_P[\mathbb{G}_{\mathbb{Q}}])$ .

Suppose  $\{(x_1, y_1), (x_2, y_2)\} \in E(\mathcal{G}_P[\mathbb{G}_{\mathbb{Q}}])$ . Hence, either  $x_1 = x_2$  and  $\{y_1, y_2\} \in E(\mathcal{G}_{Q_{x_1}})$  or  $\{x_1, x_2\} \in E(\mathcal{G}_P)$ . If  $x_1 = x_2$  and  $\{y_1, y_2\} \in E(\mathcal{G}_{Q_{x_1}})$ , without loss of generality, assume that  $y_1 <_{Q_{x_1}} y_2$ , then  $(x_1, y_1) <_{P[\mathbb{Q}]} (x_2, y_2)$ , which implies that  $\{(x_1, y_1), (x_2, y_2)\} \in E(\mathcal{G}_{P[\mathbb{Q}]})$ . If  $\{x_1, x_2\} \in E(\mathcal{G}_P)$ , assume that  $x_1 <_P x_2$ , then  $(x_1, y_1) <_{P[\mathbb{Q}]} (x_2, y_2)$ , and so  $\{(x_1, y_1), (x_2, y_2)\} \in E(\mathcal{G}_{P[\mathbb{Q}]})$ . Therefore, we have  $E(\mathcal{G}_P[\mathbb{G}_{\mathbb{Q}}]) \subseteq E(\mathcal{G}_{P[\mathbb{Q}]})$ . We accomplish the proof.  $\square$

Given a group  $G$ , let  $\mathcal{C}(G)$  denote the set of all cyclic subgroups of  $G$ . Note that  $(\mathcal{C}(G), \subseteq)$  is a poset. The following result is clear from Lemma 2.10.

**Lemma 2.14** *Let  $G$  be a group. Then  $L_G/\mathcal{C}'(G)$  is isomorphic to  $(\mathcal{C}(G), \subseteq)$ .*

For a group  $G$ , define  $\mathcal{I}_G$  as the graph with the vertex set  $\mathcal{C}(G)$ , and two cyclic subgroups are adjacent if one is contained in the other. Then  $\mathcal{I}_G$  is the comparability graph of the poset  $(\mathcal{C}(G), \subseteq)$ . For  $C \in \mathcal{C}(G)$ , let  $\mathcal{K}_C$  be the complete graph of order  $\varphi(|C|)$ , where  $\varphi$  is the Euler's totient function. Write  $\mathbb{K}_G = \{\mathcal{K}_C \mid C \in \mathcal{C}(G)\}$ .

**Theorem 2.15** *Given a group  $G$ , the power graph  $\mathcal{P}_G$  is isomorphic to the generalized lexicographic product  $\mathcal{I}_G[\mathbb{K}_G]$ .*

*Proof.* With reference to (1) and by Definition 2.1, for any  $i$ , the subposet  $L_G([x_{i,1}])$  is a totally ordered set, i.e., every pair of distinct elements in  $[x_{i,1}]$  are comparable. Therefore, the comparability graph  $\mathcal{G}_{L_G([x_{i,1}])}$  is the complete graph of order  $|[x_{i,1}]|$ . Since  $|[x_{i,1}]| = \varphi(|\langle x_{i,1} \rangle|)$ , we have  $\mathcal{G}_{L_G([x_{i,1}])} \simeq \mathcal{K}_{\langle x_{i,1} \rangle}$ . By Lemma 2.14, we obtain  $\mathcal{G}_{L_G/\mathcal{C}'(G)} \simeq \mathcal{I}_G$ . It follows that

$$\mathcal{G}_{L_G/\mathcal{C}'(G)}[\{\mathcal{G}_{L_G([x_{i,1}])} \mid [x_{i,1}] \in \mathcal{C}'(G)\}] \simeq \mathcal{I}_G[\mathbb{K}_G]. \quad (2)$$

By Lemma 2.8 we get  $L_G \simeq (L_G/\mathcal{C}'(G))[\{L_G([x_{i,1}]) \mid [x_{i,1}] \in \mathcal{C}'(G)\}]$ . Combining Example 1, Lemma 2.13 and (2), one has  $\mathcal{P}_G = \mathcal{G}_{L_G} \simeq \mathcal{I}_G[\mathbb{K}_G]$ , as desired.  $\square$

Since the order of  $\mathcal{I}_G[\mathbb{K}_G]$  is  $\sum_{C \in \mathcal{C}(G)} \varphi(|C|)$ , by Theorem 2.15, we get a Euler's classical formula on the finite groups. If  $G$  is cyclic, then this formula is the Euler's classical formula.

**Corollary 2.16** *Let  $G$  be a group. Then*

$$\sum_{C \in \mathcal{C}(G)} \varphi(|C|) = |G|.$$

Finally, we give a necessary and sufficient condition for two isomorphic power graphs.

**Theorem 2.17** *Let  $G_1$  and  $G_2$  be two groups. Then the followings are equivalent.*

- (i) *The power graphs  $\mathcal{P}_{G_1}$  and  $\mathcal{P}_{G_2}$  are isomorphic.*
- (ii) *There is an isomorphism  $\sigma$  from the poset  $(\mathcal{C}(G_1), \subseteq)$  to the poset  $(\mathcal{C}(G_2), \subseteq)$  such that  $|\sigma(C)| = |C|$  for each  $C \in \mathcal{C}(G_1)$ .*

*Proof.* It follows from Theorem 2.12 and Lemma 2.14 that (ii) implies (i). Suppose (i) holds. By Theorem 2.12 and Lemma 2.14, there exists an isomorphism  $\sigma$  from  $(\mathcal{C}(G_1), \subseteq)$  to  $(\mathcal{C}(G_2), \subseteq)$  such that  $\varphi(|\sigma(C)|) = \varphi(|C|)$  for each  $C \in \mathcal{C}(G_1)$ , where  $\varphi$  is the Euler's totient function. In order to prove (ii), we only need to show that  $|\sigma(C)| = |C|$  for each  $C \in \mathcal{C}(G_1)$ .

Suppose for the contradiction that there exists a cyclic subgroup  $C_0$  of  $G_1$  such that  $|\sigma(C_0)| \neq |C_0|$ . Since  $\varphi(|\sigma(C_0)|) = \varphi(|C_0|)$ , there exists a prime  $p$  such that one of  $|\sigma(C_0)|$  and  $|C_0|$  is divided by  $p$  and the other is not.

If  $p$  divides  $|C_0|$ , there exists a cyclic subgraph  $C_1$  of  $C_0$  with order  $p$ . Then

$$\varphi(|\sigma(C_1)|) = \varphi(|C_1|) = p - 1. \quad (3)$$



For  $i \in \{1, 2\}$ , let  $e_i$  denote the identity of  $G_i$ . Then  $\sigma(\langle e_1 \rangle) = \langle e_2 \rangle$ . Since there is no cyclic subgroup  $C_2$  of  $C_1$  such that  $\{e_1\} \subsetneq C_2 \subsetneq C_1$ , there is no cyclic subgroup  $C'_2$  of  $\sigma(C_1)$  such that  $\{e_2\} \subsetneq C'_2 \subsetneq \sigma(C_1)$ , which implies that  $|\sigma(C_1)|$  is a prime, and so  $|\sigma(C_1)| = p$  by (3). Note that  $\sigma(C_1) \subseteq \sigma(C_0)$ . Then  $p$  divides  $|\sigma(C_0)|$ , a contradiction. If  $p$  divides  $|\sigma(C_0)|$ , we consider the inverse isomorphism  $\sigma^{-1}$ , and similarly get a contradiction.  $\square$

By the above theorem, we get the following proposition.

**Proposition 2.18** [8, Corollary 3] *Two groups whose power graphs are isomorphic have the same numbers of elements of each order.*

### 3 Metric dimension

In this section we establish a closed formula for the metric dimension of the power graph of a group  $G$ . In Subsection 3.1, we give an equivalence relation on  $G$  and denote by  $\mathcal{U}(G)$  the set of all equivalence classes. If  $G$  is cyclic, then  $\mathcal{U}(G)$  is determined; otherwise we characterise all equivalence classes in  $\mathcal{U}(G)$  by using homogeneous sets in  $L_G$ . In Subsection 3.2, we introduce a concept named resolving involution and denote by  $W(G)$  the set of all resolving involutions of  $G$ . If  $G$  is cyclic, then  $W(G)$  is determined; otherwise, by using homogeneous sets in a subposet of  $L_G$ , we provide a necessary and sufficient condition for an involution to be a resolving involution of  $G$ . In Subsection 3.3, we establish a closed formula for  $\dim(\mathcal{P}_G)$  in terms of  $|G|$ ,  $|\mathcal{U}(G)|$  and  $|W(G)|$ . In particular, we compute the metric dimension of the power graph of a cyclic group.

#### 3.1 Equivalence classes

Given an element  $x$  in a group  $G$ , the *open neighborhood* of  $x$  in the power graph  $\mathcal{P}_G$ , denoted by  $N(x)$ , is the set  $\{y \in G \mid d_{\mathcal{P}_G}(x, y) = 1\}$ ; the *closed neighborhood* of  $x$  in  $\mathcal{P}_G$ , denoted by  $N[x]$ , is the union of  $N(x)$  and  $\{x\}$ .

For two elements  $x$  and  $y$  in a group  $G$ , define  $x \equiv y$  if  $N(x) = N(y)$  or  $N[x] = N[y]$ . Hernando et al. [20] proved that  $\equiv$  is an equivalence relation. Let  $\bar{x}$  denote the equivalence class that contains  $x$ . Write

$$\mathcal{U}(G) = \{\bar{x} \mid x \in G\}.$$

**Observation 2** Let  $x$  be an element of a group  $G$ .

- (i)  $[x] \subseteq \bar{x}$ .
- (ii)  $\bar{x} = \{y \mid y \in G, N(y) = N(x)\}$  or  $\{y \mid y \in G, N[y] = N[x]\}$ . In particular, the equivalence class  $\bar{x}$  is an independent set or a clique in  $\mathcal{P}_G$ .

A *maximal involution* of a group  $G$  is an involution  $x$  such that  $\langle x \rangle$  is a maximal cyclic subgroup of  $G$ . For  $y \in G$ , let  $o(y)$  denote the order of  $y$  in the rest of this paper.

**Lemma 3.1** *Let  $G$  be a group of order at least two. Suppose that  $x$  and  $y$  are two distinct elements in  $G$ . Then  $N(x) = N(y)$  if and only if both  $x$  and  $y$  are maximal involutions of  $G$ .*



*Proof.* If both  $x$  and  $y$  are maximal involutions, then  $G$  is noncyclic, and so  $N(x) = \{e\} = N(y)$ . Now suppose  $N(x) = N(y)$ .

If  $o(x) \geq 3$ , then  $x^{-1} \neq x$ . Note that  $N[x] = N[x^{-1}]$ . Since  $x^{-1} \in N(x)$ , we have  $x^{-1} \in N(y)$ , and so  $y \in N(x^{-1})$ , which implies that  $y \in N(x)$ , a contradiction. So  $o(x) = 2$ . Similarly, we have  $o(y) = 2$ .

If  $\langle x \rangle$  is not a maximal cyclic subgroup, there exists an element  $z$  of even order in  $G \setminus \{x\}$  such that  $\langle x \rangle \subseteq \langle z \rangle$ , which implies that  $z \in N(x)$ , and so  $z \in N(y)$ . Consequently, one gets  $\langle y \rangle \subseteq \langle z \rangle$ . Note that the involution in a cyclic group of even order is unique. Hence  $x = y$ , a contradiction. Therefore  $\langle x \rangle$  is a maximal cyclic subgroup. We obtain that  $\langle y \rangle$  is a maximal cyclic subgroup similarly.  $\square$

**Lemma 3.2** *Given a group  $G$ , let  $U$  be a homogeneous antichain or a homogeneous chain in  $L_G$ . Then  $U \subseteq \bar{u}$  for any  $u \in U$ .*

*Proof.* Pick  $x, y \in U$  and  $z \in G \setminus U$ . Since  $U$  is homogeneous, we have  $z \in N(x)$  is equivalent to  $z \in N(y)$ . Hence, if  $U$  is an antichain, then  $N(x) = N(y)$ ; if  $U$  is a chain, then  $N[x] = N[y]$ . Consequently, the desired result follows.  $\square$

Let  $G$  be a group. The following result, the proof of which is immediate from Lemmas 3.1 and 3.2, characterise equivalence classes in  $\mathcal{U}(G)$  that is an independent set with at least two vertices in  $\mathcal{P}_G$ .

**Proposition 3.3** *Suppose that  $U$  is a subset of a group  $G$  and  $|U| \geq 2$ . Then the followings are equivalent.*

- (i) *The set  $U$  is an equivalence class in  $\mathcal{U}(G)$  that is an independent set in  $\mathcal{P}_G$ .*
- (ii) *The set  $U$  consists of all maximal involutions of  $G$ .*
- (iii) *The set  $U$  is a maximal homogeneous antichain in  $L_G$ .*

Given a group  $G$ , we always use  $e$  to denote the identity in the remaining of this paper. Now we consider the equivalence class in  $\mathcal{U}(G)$  that is a clique in  $\mathcal{P}_G$ . Note that  $\bar{e}$  is always a clique in  $\mathcal{P}_G$ .

**Proposition 3.4** [7, Proposition 4] (i) *Suppose  $G$  is a cyclic group generated by  $x$ . If  $|G|$  is a prime power, then  $\bar{e} = G$ . If  $|G|$  is not a prime power, then  $\bar{e} = [e] \cup [x]$ .*  
(ii) *If  $G$  is a generalized quaternion 2-group, then  $\bar{e} = \{e, x\}$ , where  $x$  is the unique involution in  $G$ .*  
(iii) *If a group  $G$  is neither a cyclic group nor a generalized quaternion 2-group, then  $\bar{e} = \{e\}$ .*

**Proposition 3.5** [7, Proposition 5] *Let  $x$  be an element of a group  $G$ . Suppose that  $\bar{x} \neq \bar{e}$  and  $\bar{x}$  is a clique in  $\mathcal{P}_G$ . Then one of the following holds.*

- (i)  $\bar{x} = [x]$ .
- (ii) *There exist elements  $x_0, x_1, \dots, x_r$  in  $G$  with  $\langle x_0 \rangle \subsetneq \langle x_1 \rangle \subsetneq \dots \subsetneq \langle x_r \rangle$  and  $o(x_i) = p^{s+i}$  for each  $i \in \{0, 1, \dots, r\}$ , where  $p$  is a prime and  $s$  is a positive integer, such that*

$$\bar{x} = [x_0] \cup [x_1] \cup \dots \cup [x_r].$$

For a cyclic group  $G$ , the set  $\mathcal{U}(G)$  is determined in the following result.

**Proposition 3.6** *Suppose a cyclic group  $G = \langle x \rangle$ .*

- (i) *If  $o(x)$  is a prime power, then  $\mathcal{U}(G) = \{G\}$ .*
- (ii) *If  $o(x)$  is not a prime power, then*

$$\mathcal{U}(G) = (\mathcal{C}'(G) \setminus \{[e], [x]\}) \cup \{[e] \cup [x]\}.$$

*Proof.* (i) It is immediate from Proposition 3.4.

(ii) By Proposition 3.4, we only need to show that  $\overline{y} = [y]$  for any  $y \in G \setminus ([e] \cup [x])$ . If  $\overline{y} \neq [y]$ , by Proposition 3.5, we have  $o(y) = p^s$  and there exists an element  $z \in G \setminus [y]$  with  $o(z) = p^t$  such that  $[y] \cup [z] \subseteq \overline{y}$ , where  $p$  is a prime and  $s, t$  are two distinct positive integers. Choose a prime divisor  $q$  of  $|G|$  with  $q \neq p$ . Hence, there exists an element  $u \in G$  with  $o(u) = p^{\max\{s, t\}}q$ , which implies that  $u \in N[y] \setminus N[z]$  or  $u \in N[z] \setminus N[y]$ , and so  $N[y] \neq N[z]$ , a contradiction.  $\square$

**Lemma 3.7** *Let  $x$  be an element of a noncyclic group  $G$ . If  $\overline{x}$  is a clique in  $\mathcal{P}_G$ , then  $\overline{x}$  is a homogeneous chain in  $L_G$ .*

*Proof.* If  $\overline{x} = \overline{e}$ , then  $G$  is a generalized quaternion 2-group and  $\overline{x} = \{e, x_0\}$  by Propositions 3.4, where  $x_0$  is the unique involution. For any  $y \in G \setminus \overline{x}$ , we have  $x_0 = y^{\frac{o(y)}{2}}$  and  $e = y^{o(y)}$ , which implies that  $x_0 \prec y$  and  $e \prec y$ . So  $\overline{x}$  is a homogeneous chain in  $L_G$ . Now suppose  $\overline{x} \neq \overline{e}$ . Then (i) or (ii) in Proposition 3.5 holds. If (i) holds, then  $\overline{x}$  is a homogeneous chain in  $L_G$  by Lemma 2.9. Suppose (ii) holds. Then there exist elements  $x_1$  and  $x_2$  in  $\overline{x}$  with  $o(x_1) = p^s$  and  $o(x_2) = p^t$ , where  $p$  is a prime and  $s < t$ , such that  $\overline{x} = \{y \mid \langle x_1 \rangle \subseteq \langle y \rangle \subseteq \langle x_2 \rangle\}$ . If  $\overline{x}$  is not a homogeneous chain in  $L_G$ , there exist elements  $z \in G \setminus \overline{x}$  and  $y_1, y_2 \in \overline{x}$  such that  $y_1 \prec z \prec y_2$ , then  $\langle x_1 \rangle \subseteq \langle y_1 \rangle \subsetneq \langle z \rangle \subsetneq \langle y_2 \rangle \subseteq \langle x_2 \rangle$ , and so  $z \in \overline{x}$ , a contradiction.  $\square$

Let  $G$  be a noncyclic group. In the following two propositions, by using homogeneous sets in  $L_G$ , we characterise equivalence classes in  $\mathcal{U}(G)$  that is a clique in  $\mathcal{P}_G$ . The proof of Proposition 3.8 is clear from Lemmas 3.2 and 3.7, and the proof of Proposition 3.9 is immediate from Propositions 3.3 and 3.8.

**Proposition 3.8** *Suppose that  $U$  is a subset of a noncyclic group  $G$  and  $|U| \geq 2$ . Then  $U$  is an equivalence class in  $\mathcal{U}(G)$  that is a clique in  $\mathcal{P}_G$  if and only if  $U$  is a maximal homogeneous chain in  $L_G$ .*

**Proposition 3.9** *Let  $x$  be an element of a noncyclic group  $G$ . Then  $\{x\} \in \mathcal{U}(G)$  if and only if  $\{x\}$  is a maximal homogeneous chain and a maximal homogeneous antichain in  $L_G$ .*

### 3.2 Resolving involutions

We begin this subsection by a notation. For elements  $x$  and  $y$  in a group  $G$ , write

$$R\{x, y\} = \{z \mid z \in G, d_{\mathcal{P}_G}(x, z) \neq d_{\mathcal{P}_G}(y, z)\}.$$

**Observation 3** Let  $G$  be a group. Pick two distinct elements  $x$  and  $y$ .

- (i) Any resolving set of  $\mathcal{P}_G$  intersects  $R\{x, y\}$  nonempty.
- (ii) The equation  $\bar{x} = \bar{y}$  holds if and only if  $R\{x, y\} = \{x, y\}$ .
- (iii) If there exists an element  $z \in R\{x, y\} \setminus \{x, y\}$ , then  $\bar{z} \subseteq R\{x, y\}$ .

A *resolving involution* of a group  $G$  is an involution  $w$  satisfies that there exist two elements  $x, y \in G \setminus \bar{w}$  with  $R\{x, y\} = \{x, y, w\}$ . Let  $W(G)$  denote the set of all resolving involutions of  $G$ . For each  $w \in W(G)$ , fix two elements  $x_w$  and  $y_w$  such that  $R\{x_w, y_w\} = \{x_w, y_w, w\}$ .

**Observation 4** Suppose that  $w$  is a resolving involution of a group  $G$ .

- (i) Then  $\bar{w} = \{w\}$ .
- (ii) Then  $\bar{x_w}, \bar{y_w}$  and  $\bar{w}$  are pairwise distinct.
- (iii) For each pair  $(x, y) \in \bar{x_w} \times \bar{y_w}$ , we have  $R\{x, y\} = \{x, y, w\}$ .

**Lemma 3.10** Let  $w$  be a resolving involution of a group  $G$ . Then  $\langle x_w \rangle \subseteq \langle y_w \rangle$  or  $\langle y_w \rangle \subseteq \langle x_w \rangle$ .

*Proof.* Suppose for the contrary that  $\langle x_w \rangle \not\subseteq \langle y_w \rangle$  and  $\langle y_w \rangle \not\subseteq \langle x_w \rangle$ . Then  $x_w$  and  $y_w$  are not adjacent in  $\mathcal{P}_G$ . Hence  $[x_w] \cup [y_w] \subseteq R\{x_w, y_w\} = \{x_w, y_w, w\}$ . By Observations 2 and 4, we have  $[x_w] = \{x_w\}$  and  $[y_w] = \{y_w\}$ , which implies that  $o(x_w) = o(y_w) = 2$ , and so  $w \notin R\{x_w, y_w\}$ , a contradiction.  $\square$

**Lemma 3.11** Let  $w$  be a resolving involution of a group  $G$ . Then there exists a cyclic subgroup  $C$  of  $G$  such that  $w$  is the resolving involution of  $C$ .

*Proof.* It suffices to show that there exists a cyclic subgroup  $C$  of  $G$  such that  $\{x_w, y_w, w\} \subseteq C$ . By Lemma 3.10, without loss of generality, assume that  $\langle x_w \rangle \subseteq \langle y_w \rangle$ . Hence, we only need to consider  $w \notin \langle y_w \rangle$ . Since  $w \in R\{x_w, y_w\}$ , we have  $x_w \in \langle w \rangle$ , which implies that  $x_w = e$ .

*Claim 1.* For any  $z \in G \setminus \{w\}$ , we have  $\langle z \rangle \subseteq \langle y_w \rangle$  or  $\langle y_w \rangle \subsetneq \langle z \rangle$ . In fact, if  $z \in G \setminus \{e, y_w, w\}$ , then  $z \notin R\{e, y_w\}$ , which implies that  $z$  is adjacent to  $y_w$  in  $\mathcal{P}_G$ . Hence, Claim 1 is valid.

Write

$$\mathcal{A} = \{\langle z \rangle \mid z \in G \setminus \{w\}, \langle y_w \rangle \subsetneq \langle z \rangle\}.$$

If  $\mathcal{A} = \emptyset$ , by Claim 1, we have  $\langle y_w \rangle = G \setminus \{w\}$ , which implies that  $(|G| - 1)$  is a divisor of  $|G|$ , a contradiction. So  $\mathcal{A} \neq \emptyset$ .

*Claim 2.* For any  $\langle z \rangle \in \mathcal{A}$ , if  $w \notin \langle z \rangle$ , then  $o(z)$  is a prime power. In fact, if  $o(z)$  is not a prime power, since  $o(y_w)$  divides  $o(z)$  and  $o(y_w) \neq o(z)$ , there exists a divisor  $m$  of  $o(z)$  such that  $m$  does not divide  $o(y_w)$  and  $o(y_w)$  does not divide  $m$ . Pick  $z_0 \in \langle z \rangle$  with  $o(z_0) = m$ . Then  $z_0 \neq w$ ,  $\langle z_0 \rangle \not\subseteq \langle y_w \rangle$  and  $\langle y_w \rangle \not\subseteq \langle z_0 \rangle$ , contrary to Claim 1. Hence, Claim 2 holds.

Suppose that  $w \notin \langle z \rangle$  for any  $\langle z \rangle \in \mathcal{A}$ . By Claim 2, it is clear that  $o(y_w)$  is a prime power. Write  $o(y_w) = p^s$ , where  $p$  is a prime and  $s$  is a positive integer. By Claims 1 and 2, the following claim is valid.

*Claim 3.* For any  $z \in G \setminus \{w\}$ , we get  $o(z) = p^i$  for some nonnegative integer  $i$ .

*Claim 4.* The subgroup of order  $p$  that is contained in  $G \setminus \{w\}$  is unique. In fact, the subgroup of order  $p$  in  $\langle y_w \rangle$  is unique, which we denote by  $P$ . If there exists two subgroups  $P$  and  $Q$  of order  $p$  such that  $P \cup Q \subseteq G \setminus \{w\}$ , then  $Q \cap \langle y_w \rangle = \{e\}$ , contrary to Claim 1. Hence, Claim 4 holds.

Write  $m_i = |\{x \mid x \in G \setminus \{w\}, o(x) = p^i\}|$ . Let  $t$  be the maximum number of  $i$  such that  $m_i \neq 0$ . By Claim 3, we have

$$\sum_{i=0}^t m_i = |G| - 1. \quad (4)$$

Since  $\varphi(p^i)$  divides  $m_i$ , the prime  $p$  divides  $m_i$  for  $i \in \{2, \dots, t\}$ , which implies that  $p$  divides  $|G| - 1 - m_0 - m_1$  by (4). It is clear that  $p$  divides  $|G|$  and  $m_0 = 1$ . So  $p$  divides  $m_1 + 2$ . By Claim 4, we have  $m_1 = p - 1$ , which implies that  $p$  divides  $p + 1$ , a contradiction.

Therefore, there exists a cyclic subgroup  $\langle z \rangle \in \mathcal{A}$  with  $w \in \langle z \rangle$ , which implies that  $\{x_w, y_w, w\} \subseteq \langle z \rangle$ .  $\square$

For a cyclic group  $G$ , the set  $W(G)$  is determined in the following result.

**Proposition 3.12** *Let  $w$  be the involution of a cyclic group  $G$ .*

- (i) *If  $w$  is a resolving involution, then  $|G| = 2p^m$  or  $2^m p$  for some positive integer  $m$  and odd prime  $p$ .*
- (ii) *If  $|G| = 2p^m$ , then  $w$  is a resolving involution and*

$$\{o(x_w), o(y_w)\} \in \{\{1, p\}, \{2p^m, p\}\}.$$

- (iii) *If  $|G| = 2^m p$  and  $m \geq 2$ , then  $w$  is a resolving involution and*

$$\{o(x_w), o(y_w)\} = \{2p, p\}.$$

*Proof.* (i) Write  $G' = G \setminus \{x_w, y_w, w\}$ . Since  $R\{x_w, y_w\} = \{x_w, y_w, w\}$ , we get the following claim.

*Claim 1.* For any  $z \in G'$ , we have  $d_{P_G}(z, x_w) = d_{P_G}(z, y_w)$ .

Since  $w \in R\{x_w, y_w\}$ , in  $\mathcal{P}_G$  one of  $x_w$  and  $y_w$  is adjacent to  $w$  and the other is not. Without loss of generality, assume that  $x_w$  and  $w$  are adjacent. Then  $y_w$  and  $w$  are not adjacent. Hence, the following claim is valid.

*Claim 2.* The number  $o(y_w)$  is odd and  $o(y_w) \geq 3$ .

Write  $|G| = 2^{s_0} p_1^{s_1} \cdots p_t^{s_t}$ , where  $p_1, \dots, p_t$  are odd primes and  $s_0, s_1, \dots, s_t$  are positive integers. Now we divide our proof into two cases.

*Case 1.*  $\overline{x_w} = \bar{e}$ . Then  $x_w$  is adjacent to any element of  $G'$  in  $\mathcal{P}_G$ . If  $s_0 \geq 2$ , there exists an element  $z_0 \in G'$  of order 4, then  $z_0$  and  $y_w$  are adjacent by Claim 1, and so 4 divides  $o(y_w)$  or  $o(y_w)$  divides 4, contrary to Claim 2. Hence  $s_0 = 1$ . If  $t \geq 2$ , then there exist elements  $z_1$  and  $z_2$  in  $G'$  of order  $2p_1$  and  $2p_2$ , respectively. By Claim 1, both  $z_1$  and  $z_2$  are adjacent to  $y_w$ . It follows from Claim 2 that  $o(y_w) = p_1 = p_2$ , a contradiction. Hence  $t = 1$  and  $o(y_w) = p_1$ . So  $|G| = 2p_1^{s_1}$ .

*Case 2.*  $\overline{x_w} \neq \bar{e}$ . Then  $o(x_w)$  is even. Write  $o(x_w) = 2^{i_0} p_{j_1}^{i_1} \cdots p_{j_l}^{i_l}$ , where  $\{j_1, \dots, j_l\} \subseteq \{1, \dots, t\}$  and  $1 \leq i_k \leq s_k$  for  $k \in \{0, 1, \dots, l\}$ . Similar to Case 1,

we get  $i_0 = 1$ ,  $l = 1$  and  $o(y_w) = p_{j_1}$ . So  $o(x_w) = 2p_{j_1}^{i_1}$ . If  $t \geq 2$ , there exists an element  $z_3 \in G'$  such that  $o(z_3) = p_{j_1}q$  for some prime  $q \in \{p_1, \dots, p_t\} \setminus \{p_{j_1}\}$ , then  $z_3$  is adjacent to  $y_w$  and not adjacent to  $x_w$ , contrary to Claim 1. Hence  $t = 1$ . Consequently, we get  $|G| = 2^{s_0}p_1^{s_1}$ ,  $o(x_w) = 2p_1^{i_1}$  and  $o(y_w) = p_1$ . If  $i_1 < s_1$ , then any element of order  $p_1^{i_1+1}$  in  $G'$  is adjacent to  $y_w$  and not adjacent to  $x_w$ , contrary to Claim 1. Therefore  $o(x_w) = 2p_1^{s_1}$ . If  $s_1 \geq 2$ , then any element of order  $p_1^2$  in  $G'$  is adjacent to  $y_w$  and not adjacent to  $x_w$ , contrary to Claim 1. Hence  $s_1 = 1$ , and so  $|G| = 2^{s_0}p_1$ .

(ii) Suppose  $y$  is an element of  $G$  with  $o(y) = p$ . Then  $R\{e, y\} = \{e, y, w\}$ , which implies that  $w$  is a resolving involution of  $G$ . Combining Proposition 3.4 and the proof of (i), we have  $\{o(x_w), o(y_w)\} \in \{\{1, p\}, \{2p^m, p\}\}$ .

(iii) Suppose that  $x_1$  and  $x_2$  are two elements of  $G$  with  $o(x_1) = 2p$  and  $o(x_2) = p$ . Then  $R\{x_1, x_2\} = \{x_1, x_2, w\}$ , which implies that  $w$  is a resolving involution of  $G$ . It follows from the proof of (i) that  $\{o(x_w), o(y_w)\} = \{2p, p\}$ .  $\square$

**Lemma 3.13** [18, Theorem 5.4.10. (ii)] *Let  $p$  be a prime. If  $G$  is a  $p$ -group which has a unique minimal subgroup of order  $p$ , then  $G$  is either a cyclic group or a generalized quaternion group.*

In the rest of this subsection, we consider the resolving involutions of a noncyclic group.

**Proposition 3.14** *Let  $w$  be a resolving involution of a noncyclic group  $G$ . Suppose  $o(x_w) \leq o(y_w)$ . Then the follows hold.*

- (i)  $\langle x_w \rangle \cup \langle w \rangle \subseteq \langle y_w \rangle$ .
- (ii) *There exists an odd prime divisor  $p$  of  $|G|$  such that  $(o(x_w), o(y_w)) = (p, 2p^m)$  for some positive integer  $m$ .*

*Proof.* In order to prove (i) and (ii), combining Lemma 3.11 and Proposition 3.12, we only need to show that  $(o(x_w), o(y_w)) \neq (1, q)$  for any odd prime  $q$ . Suppose for the contrary that  $x_w = e$  and  $o(y_w) = q$  for some odd prime  $q$ . Since  $R\{e, y_w\} = \{e, y_w, w\}$ , each element in  $G \setminus \{w, y_w\}$  is adjacent to  $y_w$  in  $\mathcal{P}_G$ , which implies that  $y_w \in \langle z \rangle$  for any  $z \in G \setminus \{e, w\}$ . Hence, the following claims are valid.

*Claim 1.* All prime divisors of  $|G|$  are 2 and  $q$ .

*Claim 2.* The group  $G$  contains a unique involution, which is  $w$ , and a unique subgroup of order  $q$ , which is  $\langle y_w \rangle$ .

*Claim 3.* There is no element of order 4 in  $G$ .

By Claims 2 and 3, the subgroup  $\langle w \rangle$  is a unique Sylow 2-subgroup of  $G$ , and so  $\langle w \rangle$  is normal in  $G$ . By Claim 1, we have  $|G| = 2q^n$  for some positive integer  $n$ . By Claim 2 and Lemma 3.13, a Sylow  $q$ -subgroup  $Q$  of  $G$  is isomorphic to the cyclic group of order  $q^n$ . Since the index of  $Q$  in  $G$  is 2, the Sylow  $q$ -subgroup  $Q$  is normal in  $G$ . Consequently, the group  $G$  is isomorphic to  $\langle w \rangle \times Q$ , which is isomorphic to the cyclic group of order  $2q^n$ , a contradiction.  $\square$

Given a noncyclic group  $G$ , by using the homogeneous set in a subposet of  $L_G$ , we provide a necessary and sufficient condition for an involution to be a resolving involution of  $G$ .

**Proposition 3.15** *Let  $w$  be an involution of a noncyclic group  $G$ . Then  $w$  is a resolving involution of  $G$  if and only if there exists a cyclic subgroup  $C$  of  $G$  such that the following conditions hold.*

- (i)  $|C| = 2p^m$  for some odd prime  $p$  and positive integer  $m$ .
- (ii)  $w \in C$ .
- (iii) *The set  $C \setminus \langle w \rangle$  is homogeneous in the subposet  $L_G(G \setminus \{w\})$ .*

*Proof.* Suppose  $w$  is a resolving involution of  $G$ . Without loss of generality, assume that  $o(x_w) \leq o(y_w)$ . By Proposition 3.14, we have  $\langle x_w \rangle \cup \langle w \rangle \subseteq \langle y_w \rangle$  and  $(o(x_w), o(y_w)) = (p, 2p^m)$  for some odd prime  $p$  and positive integer  $m$ . Let  $C = \langle y_w \rangle$ . Then (i) and (ii) hold. Now we prove (iii).

For each  $x \in C \setminus \langle w \rangle$ , since  $\langle x_w \rangle \subseteq \langle x \rangle \subseteq \langle y_w \rangle$ , we have

$$x_w \preceq x \preceq y_w. \quad (5)$$

Pick any  $z \in (G \setminus \{w\}) \setminus (C \setminus \langle w \rangle)$ . Then  $z = e$  or  $z \in G \setminus C$ . If  $z = e$ , then  $z \preceq x$  for each  $x \in C \setminus \langle w \rangle$ . In the following two cases, suppose  $z \in G \setminus C$ .

*Case 1.*  $C \subseteq \langle z \rangle$ . Then  $y_w \preceq z$ . For each  $x \in C \setminus \langle w \rangle$ , by (5), we have  $x \preceq z$ .

*Case 2.*  $C \not\subseteq \langle z \rangle$ . If there exists an element  $x_1 \in C \setminus \langle w \rangle$  such that  $z \preceq x_1$ , then  $z \preceq y_w$  by (5). So  $z \in C$ , a contradiction. If there exists an element  $x_2 \in C \setminus \langle w \rangle$  such that  $x_2 \preceq z$ , then  $x_w \preceq z$  by (5). Hence  $z$  is adjacent to  $x_w$  in  $\mathcal{P}_G$ . Since  $z \notin \{x_w, y_w, w\} = R\{x_w, y_w\}$ , elements  $z$  and  $y_w$  are adjacent, which implies that  $z \in C$  or  $C \subseteq \langle z \rangle$ , a contradiction. Therefore, for each  $x \in C \setminus \langle w \rangle$ , elements  $z$  and  $x$  are incomparable in  $L_G(G \setminus \{w\})$ .

Hence (iii) holds.

Conversely, if there exists a cyclic subgroup  $C$  of  $G$  such that (i), (ii) and (iii) hold. Write  $C = \langle y \rangle$ . By (i) and (ii), we get  $o(y) = 2p^m$  and  $w = y^{p^m}$ , where  $p$  is an odd prime and  $m$  is a positive integer. Consider these two vertices  $y$  and  $y^{2p^{m-1}}$  in  $\mathcal{P}_G$ . By (iii), any vertex in  $(G \setminus \{w\}) \setminus (\langle y \rangle \setminus \langle w \rangle)$  is adjacent to both or neither of them. Note that each vertex in  $\langle y \rangle \setminus \{w, y, y^{2p^{m-1}}\}$  is adjacent to both of them. Hence, we have  $R\{y, y^{2p^{m-1}}\} = \{w, y, y^{2p^{m-1}}\}$ . It follows that  $w$  is a resolving involution of  $G$ .  $\square$

The following lemma is useful for the next subsection.

**Lemma 3.16** *Let  $u$  and  $v$  be two distinct resolving involutions of a group  $G$ . Then  $\{\overline{x_u}, \overline{y_u}, \overline{u}\} \cap \{\overline{x_v}, \overline{y_v}, \overline{v}\} = \emptyset$ .*

*Proof.* Without loss of generality, assume that  $o(x_u) \leq o(y_u)$  and  $o(x_v) \leq o(y_v)$ . Since  $G$  has at least two involutions, we know that  $G$  is noncyclic. By Proposition 3.14, we get

$$\langle x_w \rangle \cup \langle w \rangle \subseteq \langle y_w \rangle \quad \text{and} \quad (o(x_w), o(y_w)) \in \{(p, 2p^m) \mid p \text{ is an odd prime}, m \geq 1\},$$

where  $w \in \{u, v\}$ . Then  $\langle y_u \rangle \not\subseteq \langle y_v \rangle$  and  $\langle y_v \rangle \not\subseteq \langle y_u \rangle$ . So  $\overline{y_u} \neq \overline{y_v}$ . If  $\overline{x_u} = \overline{x_v}$ , then  $y_v$  is adjacent to  $x_u$  in  $\mathcal{P}_G$ , which implies that  $y_v \in R\{x_u, y_u\}$ , a contradiction. Hence  $\overline{x_u} \neq \overline{x_v}$ . It follows that  $\{\overline{x_u}, \overline{y_u}, \overline{u}\} \cap \{\overline{x_v}, \overline{y_v}, \overline{v}\} = \emptyset$ , as desired.  $\square$

### 3.3 Formula

In this subsection, we shall establish a closed formula for the metric dimension of the power graph of a group. As an application, we compute  $\dim(\mathcal{P}_{Z_n})$ , where  $Z_n$  is a cyclic group of order  $n$ . We begin by some lemmas.

**Lemma 3.17** *Let  $G$  be a group. Suppose that  $S$  is a resolving set of  $\mathcal{P}_G$  and  $\bar{z} \in \mathcal{U}(G)$ . Then  $|S \cap \bar{z}| \geq |\bar{z}| - 1$ .*

*Proof.* If  $|S \cap \bar{z}| \leq |\bar{z}| - 1$ , there exist two distinct elements  $z_1, z_2 \in \bar{z}$  such that  $S \cap \{z_1, z_2\} = \emptyset$ . Since  $\bar{z}_1 = \bar{z}_2 = \bar{z}$ , by Observation 3 we have  $R\{z_1, z_2\} = \{z_1, z_2\}$  and  $S \cap R\{z_1, z_2\} \neq \emptyset$ , a contradiction.  $\square$

**Lemma 3.18** *Given a group  $G$ , we have  $\dim(\mathcal{P}_G) \geq |G| - |\mathcal{U}(G)| + |W(G)|$ .*

*Proof.* Suppose that  $S$  is a resolving set of  $\mathcal{P}_G$  with size  $\dim(\mathcal{P}_G)$ . If  $W(G) = \emptyset$ , by Lemma 3.17 we get

$$\dim(\mathcal{P}_G) = |S| = \sum_{\bar{z} \in \mathcal{U}(G)} |S \cap \bar{z}| \geq \sum_{\bar{z} \in \mathcal{U}(G)} (|\bar{z}| - 1) = |G| - |\mathcal{U}(G)|.$$

Now suppose  $W(G) \neq \emptyset$ . For each  $w \in W(G)$ , by Observations 4 and Lemma 3.17, we get

$$|S \cap (\bar{w} \cup \bar{x}_w \cup \bar{y}_w)| \geq |\bar{w}| - 1 + |\bar{x}_w| - 1 + |\bar{y}_w| - 1 + 1 = |\bar{x}_w| + |\bar{y}_w| - 1,$$

which implies that

$$\sum_{w \in W(G)} (|S \cap \bar{w}| + |S \cap \bar{x}_w| + |S \cap \bar{y}_w|) \geq \sum_{w \in W(G)} (|\bar{x}_w| + |\bar{y}_w|) - |W(G)|. \quad (6)$$

Write  $\mathcal{W}(G) = \bigcup_{w \in W(G)} \{\bar{w}, \bar{x}_w, \bar{y}_w\}$ . Combining Lemma 3.16 and (6), we have

$$\sum_{\bar{z} \in \mathcal{W}(G)} |S \cap \bar{z}| \geq \sum_{w \in W(G)} (|\bar{x}_w| + |\bar{y}_w|) - |W(G)| = \sum_{\bar{z} \in \mathcal{W}(G)} |\bar{z}| - 2|W(G)|. \quad (7)$$

By (7) and Lemma 3.17, we get

$$\begin{aligned} \dim(\mathcal{P}_G) = |S| &= \sum_{\bar{z} \in \mathcal{W}(G)} |S \cap \bar{z}| + \sum_{\bar{z} \in \mathcal{U}(G) \setminus \mathcal{W}(G)} |S \cap \bar{z}| \\ &\geq \sum_{\bar{z} \in \mathcal{W}(G)} |\bar{z}| - 2|W(G)| + \sum_{\bar{z} \in \mathcal{U}(G) \setminus \mathcal{W}(G)} (|\bar{z}| - 1) \\ &= |G| - 2|W(G)| - (|\mathcal{U}(G)| - |\mathcal{W}(G)|). \end{aligned}$$

Since  $|\mathcal{W}(G)| = 3|W(G)|$ , our desired result follows.  $\square$

We use  $\Psi$  to denote the set of noncyclic groups  $G$  satisfying that there exists an odd prime  $p$  such that the following three conditions hold.

- (C1) The prime divisors of  $|G|$  are 2 and  $p$ .
- (C2) The subgroup of order  $p$  is unique.
- (C3) There is no element of order 4 in  $G$ .
- (C4) Each involution of  $G$  is contained in a cyclic subgroup of order  $2p$ .



**Example 2** Let  $Z_n$  denote the cyclic group of order  $n$ . If  $m \geq 2, n \geq 1$  and  $p$  is an odd prime, then

$$\overbrace{Z_2 \times \cdots \times Z_2}^m \times Z_{p^n} \in \Psi.$$

**Proposition 3.19** Suppose  $G \in \Psi$ . Then  $|G| = 2^m p^n$  for some positive integers  $m, n$  and odd prime  $p$ . Moreover, the Sylow 2-subgroup is an elementary abelian 2-group and the Sylow  $p$ -subgroup is a cyclic group.

*Proof.* The condition (C1) implies that  $|G| = 2^m p^n$  for some positive integers  $m, n$  and odd prime  $p$ . By (C3), the Sylow 2-subgroup is an elementary abelian 2-group. It follows from Lemma 3.13 and (C2) that the Sylow  $p$ -subgroup is cyclic.  $\square$

**Lemma 3.20** Let  $G$  be a noncyclic group. Then  $G \in \Psi$  if and only if there is a nonidentity element  $x$  of  $G$  such that the following conditions hold.

- (i) All elements in  $R\{e, x\} \setminus \{e, x\}$  are involutions.
- (ii) There exist  $r - 3$  involutions in  $R\{e, x\} \setminus \{e, x\}$  which are not maximal involutions of  $G$ , where  $r = \max\{|R\{e, x\}|, 4\}$ .

*Proof.* Suppose  $G \in \Psi$ . Pick an element  $x \in G$  with  $o(x) = p$ , where  $p$  is an odd prime and  $p$  divides  $|G|$ . For any element  $y \in G$  with  $o(y) \geq 3$ , by (C1) and (C3), the prime  $p$  divides  $o(y)$ , which implies that  $\langle x \rangle \subseteq \langle y \rangle$  by (C2), and so  $y \notin R\{e, x\}$ . Hence (i) holds. The condition (C4) implies that (ii) holds.

Conversely, suppose that there is a nonidentity element  $x$  of  $G$  such that (i) and (ii) hold. Write

$$R_0 = R\{e, x\} \setminus \{e, x\}, \quad R_1 = \{z \mid z \in R_0, z \text{ is not a maximal involution of } G\}.$$

We claim that, for any  $z \in R_1$ , we have  $z \notin \langle x \rangle$  and there exists an element  $z' \in G$  such that  $\langle z \rangle \cup \langle x \rangle \subseteq \langle z' \rangle$ . In fact, for any  $z \in R_1$ , since  $R_1 \subseteq R_0 \subseteq R\{e, x\}$ , we have  $z \notin \langle x \rangle$ . By (i) there exists an element  $z' \in G \setminus R_0$  such that  $\langle z \rangle \subsetneq \langle z' \rangle$ . Since  $\langle z' \rangle \not\subseteq \langle x \rangle$ , we have  $\langle x \rangle \subseteq \langle z' \rangle$ . Hence, our claim is valid.

By (ii) we get  $R_1 \neq \emptyset$ . Pick  $z_0 \in R_1$ . By (i) we have  $o(z_0) = 2$ . By the claim, we have  $z_0 \notin \langle x \rangle$  and there is an element  $z'_0$  such that  $\langle z_0 \rangle \cup \langle x \rangle \subseteq \langle z'_0 \rangle$ , which implies that  $o(x)$  is odd,  $o(z'_0)$  is even and  $o(x)$  divides  $o(z'_0)$ . If  $o(x)$  is not a prime, there is an even number  $m$  with  $2 < m < o(z'_0)$  such that  $m$  divides  $o(z'_0)$  and  $o(x)$  does not divide  $m$ , which implies that any element of order  $m$  in  $\langle z'_0 \rangle$  is in  $R_0$ , contrary to (i). Hence  $o(x)$  is an odd prime.

Write  $p = o(x)$ . Then  $p$  is an odd prime. Hence, for any  $x' \in G \setminus R\{e, x\}$ , we get  $x \in \langle x' \rangle$ . Therefore, the condition (C1), (C2) and (C3) hold. Note that  $R_0$  consists of all involutions in  $G$ . In order to prove (C4), we only need to prove  $R_0 = R_1$ .

If  $|R\{e, x\}| \leq 3$ , then  $|R\{e, x\}| = 3$  and  $|R_0| = 1$ , which implies that  $R_0 = R_1$  by (ii). Now suppose  $|R\{e, x\}| \geq 4$ . By (ii), we have

$$0 \leq |R_0| - |R_1| \leq 1. \tag{8}$$

Write  $m_i = |\{g \mid g \in G, o(g) = p^i\}|$  and  $n_i = |\{g \mid g \in G, o(g) = 2p^i\}|$ . Let  $s$  and  $t$  be the maximum numbers of  $i$  such that  $m_i \neq 0$  and  $n_i \neq 0$ , respectively. By (C1) and (C3), we have

$$\sum_{i=0}^s m_i + \sum_{i=0}^t n_i = |G|. \quad (9)$$

Since  $\varphi(p^i)$  divides  $m_i$  and  $\varphi(2p^i)$  divides  $n_i$ , the prime  $p$  divides  $m_i$  and  $n_i$  for  $i \geq 2$ , which implies that  $p$  divides  $m_0 + m_1 + n_0 + n_1$  by (9). It is clear that  $m_0 = 1$ ,  $n_0 = |R_0|$  and  $n_1 = |R_1|(p-1)$ . By (C2), we have  $m_1 = p-1$ . So  $p$  divides  $|R_0| + |R_1|(p-1)$ . It follows from (8) that  $R_0 = R_1$ .  $\square$

**Lemma 3.21** *For any  $G \in \Psi$ , we have  $\dim(\mathcal{P}_G) \geq |G| - |\mathcal{U}(G)| + 1$ .*

*Proof.* By Lemma 3.20, there exists a nonidentity element  $x \in G$  such that  $R\{e, x\} \setminus \{e, x\}$  is a collection of involutions. Write  $R_0 = R\{e, x\} \setminus \{e, x\}$ . For each  $w \in R_0$ , since there is no element of order 4, by Proposition 3.5 we get  $\overline{w} = [w] = \{w\}$ . Let

$$\mathcal{U} = \mathcal{U}(G), \quad \mathcal{U}_1 = \{\overline{w} \mid w \in R_0\} \cup \{\overline{e}, \overline{x}\} \quad \text{and} \quad A = \bigcup_{\overline{z} \in \mathcal{U}_1} \{\overline{z}\}.$$

Suppose  $S$  is a resolving set of  $\mathcal{P}_G$  with size  $\dim(\mathcal{P}_G)$ . By Lemma 3.17, one gets

$$|S \cap A| \geq |\overline{x}| - 1 + |\overline{e}| - 1 + 1 = |\overline{x}| + |\overline{e}| - 1 = |A| - |R_0| - 1. \quad (10)$$

Since  $|\mathcal{U}_1| = |R_0| + 2$ , by Lemma 3.17 and (10), we have

$$\begin{aligned} \dim(\mathcal{P}_G) = |S| &= |S \cap (\bigcup_{\overline{z} \in \mathcal{U}} \overline{z})| = |S \cap A| + \sum_{\overline{z} \in \mathcal{U} \setminus \mathcal{U}_1} |S \cap \overline{z}| \\ &\geq |A| - |R_0| - 1 + \sum_{\overline{z} \in \mathcal{U} \setminus \mathcal{U}_1} (|\overline{z}| - 1) \\ &= |G| - |\mathcal{U}| + 1, \end{aligned}$$

as desired.  $\square$

Now we give a closed formula for the metric dimension of the power graph of a group.

**Theorem 3.22** *Let  $G$  be a group.*

- (i) *If  $G \in \Psi$ , then  $\dim(\mathcal{P}_G) = |G| - |\mathcal{U}(G)| + 1$ .*
- (ii) *If  $G \notin \Psi$ , then  $\dim(\mathcal{P}_G) = |G| - |\mathcal{U}(G)| + |W(G)|$ .*

*Proof.* Write  $\mathcal{U}(G) = \{\overline{x_1}, \dots, \overline{x_t}\}$  and

$$X = G \setminus \{x_1, \dots, x_t\},$$

where  $t = |\mathcal{U}(G)|$ . For any two distinct elements  $u_1$  and  $u_2$  in  $G$ , write

$$R_0\{u_1, u_2\} = R\{u_1, u_2\} \setminus \{u_1, u_2\}.$$

*Claim 1.* If there is an element of order at least three in  $R_0\{u_1, u_2\}$ , then  $X \cap R_0\{u_1, u_2\} \neq \emptyset$ . In fact, if  $z_0 \in R_0\{u_1, u_2\}$  and  $o(z_0) \geq 3$ , by Observations 2 and 3, we have  $[z_0] \subseteq \overline{z_0} \subseteq R_0\{u_1, u_2\}$ , and so  $[z_0] \setminus \{x_1, \dots, x_t\} \subseteq X \cap R_0\{u_1, u_2\}$ . Since  $|[z_0]| = \varphi(o(z_0)) \geq 2$ , one gets  $[z_0] \setminus \{x_1, \dots, x_t\} \neq \emptyset$ . Consequently, Claim 1 is valid.

*Claim 2.* If all elements in  $R_0\{u_1, u_2\}$  are involutions, then  $\langle u_1 \rangle \subsetneq \langle u_2 \rangle$  or  $\langle u_2 \rangle \subsetneq \langle u_1 \rangle$ . If  $\langle u_1 \rangle = \langle u_2 \rangle$ , then  $\overline{u_1} = \overline{u_2}$ , and so  $R_0\{u_1, u_2\} = \emptyset$ , a contradiction. Suppose that  $u_1$  and  $u_2$  are not adjacent in  $\mathcal{P}_G$ . On one hand, for any  $z \in R_0\{u_1, u_2\}$ , we conclude that  $z$  is adjacent to one of  $u_1$  and  $u_2$  and not adjacent to the other. Without loss of generality, assume that  $z$  is adjacent to  $u_1$ . Since  $o(z) = 2$  and  $u_1 \neq e$ , we have  $z \in \langle u_1 \rangle$ , and so  $o(u_1) \geq 4$ . On the other hand, since any element of  $[u_1]$  is not adjacent to  $u_2$  in  $\mathcal{P}_G$ , we have  $[u_1] \setminus \{u_1\} \subseteq R_0\{u_1, u_2\}$ , which implies that  $o(u_1) = 2$ , a contradiction. Hence, Claim 2 is valid.

(i) By Lemma 3.20, there exists a nonidentity element  $x \in G$  such that  $R_0\{e, x\}$  is a collection of involutions. Pick an element  $y_0 \in R_0\{e, x\}$ . Let

$$Y = X \cup \{y_0\}.$$

By Proposition 3.5 and (C3), we have  $\overline{y_0} = \{y_0\}$ . Then  $|Y| = |X| + 1 = |G| - |\mathcal{U}(G)| + 1$ . By Lemma 3.21, we only need to show that  $Y$  is a resolving set of  $\mathcal{P}_G$ . Pick any two distinct vertices  $u_1$  and  $u_2$  in  $G \setminus Y$ . It suffices to show that

$$Y \cap R_0\{u_1, u_2\} \neq \emptyset. \quad (11)$$

If there exists an element of order at least three in  $R_0\{u_1, u_2\}$ , by Claim 1, we have  $X \cap R_0\{u_1, u_2\} \neq \emptyset$ , which implies that (11) holds. Note that  $e \notin R_0\{u_1, u_2\}$ . Now suppose that all elements in  $R_0\{u_1, u_2\}$  are involutions. By Claim 2, without loss of generality, assume that  $\langle u_1 \rangle \subsetneq \langle u_2 \rangle$ .

In order to prove (11), we only need to show that  $y_0 \in R_0\{u_1, u_2\}$ . Suppose for the contrary that  $y_0 \notin R_0\{u_1, u_2\}$ .

Since  $\{u_1, u_2\} \subseteq G \setminus Y \subseteq \{x_1, \dots, x_t\}$ , we have  $\overline{u_1} \neq \overline{u_2}$ , which implies that  $R_0\{u_1, u_2\} \neq \emptyset$ . Pick  $u_0 \in R_0\{u_1, u_2\}$ . Then  $o(u_0) = 2$  and  $u_0$  is adjacent to one of  $u_1$  and  $u_2$  and not adjacent to the other in  $\mathcal{P}_G$ . If  $u_0$  is adjacent to  $u_2$  and not adjacent to  $u_1$  in  $\mathcal{P}_G$ , then  $\langle u_0 \rangle \subsetneq \langle u_2 \rangle$  and  $u_1 \neq e$ . Since  $u_0$  is the unique involution in the subgroup  $\langle u_2 \rangle$ , we have  $R\{u_1, u_2\} \cap \langle u_2 \rangle = \{u_1, u_2, u_0\}$ , which implies that  $u_0$  is a resolving involution of  $\langle u_2 \rangle$ . Let  $p$  be an odd prime that divides  $|G|$ . By Proposition 3.12, we have  $o(u_1) = p$  and  $o(u_2) = 2p^m$  for some positive integer  $m$ . The fact that  $o(y_0) = 2$  implies that there exists an element  $u_3$  of order  $2p$  such that  $y_0 \in \langle u_3 \rangle$  by (C4). By (C2), one has  $u_1 \in \langle u_3 \rangle$ . Since  $y_0 \neq u_0$ , we have  $y_0 \notin \langle u_2 \rangle$ , and so  $u_3 \notin \langle u_2 \rangle$ . Therefore, we get  $u_3 \in R_0\{u_1, u_2\}$ , a contradiction. Hence  $u_0$  is adjacent to  $u_1$  and not adjacent to  $u_2$  in  $\mathcal{P}_G$ , which implies that  $u_1 = e$  and  $u_0 \notin \langle u_2 \rangle$ . By (C4), there exists an element  $u_4$  of order  $2p$  such that  $u_0 \in \langle u_4 \rangle$ . Then  $u_4 \notin \langle u_2 \rangle$ . Since  $u_4 \notin R\{u_1, u_2\}$ , we have  $u_2 \in \langle u_4 \rangle$ , which implies that  $o(u_2) = p$ , and so  $y_0 \in R_0\{u_1, u_2\}$ , a contradiction.

(ii) Write

$$S = X \cup W(G).$$

Observation 4 implies that  $\overline{w} = \{w\}$  for each  $w \in W(G)$ , and so  $|S| = |G| - |\mathcal{U}(G)| + |W(G)|$ . By Lemma 3.18, we only need to show that  $S$  is a resolving set of  $\mathcal{P}_G$ . Pick

any two distinct elements  $u_1$  and  $u_2$  in  $G \setminus S$ . It suffices to show that

$$S \cap R_0\{u_1, u_2\} \neq \emptyset. \quad (12)$$

If there exists an element of order at least three in  $R_0\{u_1, u_2\}$ , by Claim 1, we have  $X \cap R_0\{u_1, u_2\} \neq \emptyset$ , which implies that (12) holds. Note that  $e \notin R_0\{u_1, u_2\}$ . Now suppose that all elements in  $R_0\{u_1, u_2\}$  are involutions. By Claim 2, without loss of generality, assume that  $\langle u_1 \rangle \subsetneq \langle u_2 \rangle$ .

If  $|R_0\{u_1, u_2\}| = 1$ , then  $R_0\{u_1, u_2\} \subseteq W(G) \subseteq S$ , and so (12) holds. Suppose  $|R_0\{u_1, u_2\}| \geq 2$ . Since  $\langle u_2 \rangle$  contains at most one involution, there exists an involution  $z_1 \in R_0\{u_1, u_2\} \setminus \langle u_2 \rangle$ . Note that  $z_1$  and  $u_2$  are not adjacent in  $\mathcal{P}_G$ . Then  $z_1$  and  $u_1$  are adjacent in  $\mathcal{P}_G$ , which implies that  $u_1 = e$  by  $z_1 \notin \langle u_1 \rangle$ . Since  $|R\{e, u_2\}| = |R_0\{u_1, u_2\}| + 2 \geq 4$  and  $G \notin \Psi$ , by Lemma 3.20, there exist two distinct maximal involutions  $v_1$  and  $v_2$  of  $G$  in  $R_0\{u_1, u_2\}$ . By Lemma 3.1, we have  $\overline{v_1} = \overline{v_2}$ , and so  $\{v_1, v_2\} \cap S \neq \emptyset$ , which implies that (12) holds.  $\square$

As a corollary, we compute the metric dimension of the power graph of a cyclic group.

**Corollary 3.23** *Suppose  $n = p_1^{r_1} \cdots p_t^{r_t}$ , where  $p_1, \dots, p_t$  are primes with  $p_1 < \cdots < p_t$ , and  $r_1, \dots, r_t$  are positive integers. Let  $Z_n$  denote the cyclic group of order  $n$ . Then*

$$\dim(\mathcal{P}_{Z_n}) = \begin{cases} n - 1, & \text{if } t = 1, \\ n - 2r_2, & \text{if } (t, p_1, r_1) = (2, 2, 1), \\ n - 2r_1, & \text{if } (t, p_1, r_2) = (2, 2, 1), \\ n + 1 - \prod_{i=1}^t (r_i + 1), & \text{otherwise.} \end{cases}$$

*Proof.* If  $t = 1$ , then  $n$  is a prime power, which implies that  $\dim(\mathcal{P}_{Z_n}) = n - 1$  by Lemma 2.6. Now suppose  $t \geq 2$ . By Propositions 3.6, we have

$$|\mathcal{U}(Z_n)| = |\mathcal{C}'(Z_n)| - 1 = |\mathcal{C}(Z_n)| - 1 = \prod_{i=1}^t (r_i + 1) - 1.$$

By Proposition 3.12, we have

$$|W(Z_n)| = \begin{cases} 1, & \text{if } (t, p_1, r_1, r_2) = (2, 2, 1, r_2) \text{ or } (2, 2, r_1, 1), \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, Theorem 3.22 (ii) implies that our desired result follows.  $\square$

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